On Constrained-Instanton Valleys

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We develop a systematic treatment for the quasi-zero modes, which play an important role in nonabelian gauge theories. It can be used to derive the analytic forms for the constrained instantons in the Yang-Mills-Higgs theory. This will automatically sum up contributions to all orders in ρ, the instanton size. We also give the analytic expressions for the instanton-antiinstanton pair, with an arbitrary relative phase. We apply the results to the computation for high energy baryon number violating cross section, as well as to Klinkhamer's new instanton.

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1. Introduction

In the Yang-Mills-Higgs theory where the Higgs mechanism breaks the classical scale invariance in the gauge sector, instantons are no longer exact solutions to the field equations[1]. Instead, they represent "the bottom of the valley" parametrized by the quasi-zero mode ρ. This valley trajectory corrresponds to low-lying configurations which still dominate the path integral. It is therefore necessary to write down an expression for it if we are to carry out a reliable semi-classical approximation for the path integral. This can be done by introducing some constraints under which instantons become solutions again. These constrained instantons are first discussed in a systematic way by Affleck[2], who uses an expansion of the supposedly small parameter ρ. Only the leading term, which is independent of the choice of contraints, is given. Terms of higher orders in ρ depend on the constraints and are difficult to evaluate. The author of ref.2 also has little to say about the pre-exponential corrections which come with any choice of constraints. These difficulties can be naturally resolved in the framework of a new systematic treatment which we shall present in this paper. In Section 2, we outline this new approach, which is far more powerful than what was called the valley method previously[3–5], and derive its general formulation. In section 3, we apply the results to the Yang-Mills-Higgs system and give the analytic form for the constrained instantons to all orders in ρ.

One of the immediate applications of this result is to the process of high energy anomalous baryon number violation. In order to include corrections from the final state particles, it is convenient to employ the optical theorem and consider the $\langle 2 \to 2 \rangle$ inclusive scattering process. The relevant configuration is therefore the instanton-antiinstanton pair, which is another quasi-zero mode by itself. Previous works[5] understandably avoided dealing directly with the Yang-Mills-Higgs system, and instead used results in the pure Yang-Mills theory[3] as an approximation. Unfortunately, not only is the contribution from the Higgs particles poorly estimated, but the gauge sector is also misrepresented by the wrong valley direction. We give the analytic expression for the Yang-Mills-Higgs instanton-antiinstanton pair in Section 4. We also clarify some other misconceptions in the literature on related issues. Multiple instanton-antiinstanton pairs can also be treated in principle. Interested readers are referred to our other paper[6].

It is rather straightforward to write down analytically our equivalent of Klinkhamer's new instanton[7] using these results. This will be done in Section 5.

2. The Quasi-zero Modes and the Semi-classical Approximation

Our objective is to evaluate path integrals in Euclidean spacetime using the semi-classical approximation. For simplicity, all spacetime indices are suppressed. We will also ignore the need for gauge fixing, which has been analysed by Yaffe[8]. The modification is straightforward and will not be included in this paper.

The semi-classical approximation for the partition function,

$$Z = \int [\mathcal{D}A] \exp(-S[A]), \tag{2.1}$$

is given by the Gaussian approximation. One first finds the minimum of $S[A]$ at A_0 by solving the field equation

$$\left. \frac{\delta S}{\delta A} \right|_{A_0} = 0, \tag{2.2}$$

makes a change of variables to $\delta A \equiv A - A_0$, and observes that the exponent now becomes

$$-S[A] = -S[A_0] - \frac{1}{2} \int d^4x \, \delta A \left. \frac{\delta^2 S}{\delta A^2} \right|_{A_0} \delta A + \mathcal{O}(\delta A^3).$$

One then proceeds by omitting the higher order terms, which are suppressed by powers of g, the coupling constant, and carrying out the Gaussian integrals. The end result is,

$$Z \sim \det^{-\frac{1}{2}} \left(\left. \frac{\delta^2 S}{\delta A^2} \right|_{A_0} \right) \exp(-S[A_0]), \tag{2.3}$$

which is the leading term in the expansion with respect to g. The vacuum expectation value of any operator $\mathcal{X}(A)$ can also be evaluated in the same manner. We have,

$$\begin{aligned} \langle \mathcal{X} \rangle &= \frac{1}{Z} \int [\mathcal{D}A] \mathcal{X}(A) \exp(-S[A]) \\ &= \mathcal{X}(A_0), \end{aligned} \tag{2.4}$$

again to the leading order in g.

Non-trivial solutions to the field equation usually come with zero modes. They have to be dealt with using the collective coordinates, which we shall denote as ω. For example, a Yang-Mills instanton is parametrized by its size and position. Adding in the global gauge

degrees of freedom, we need eight collective coordinates. The path integral (2.1) can be evaluated by inserting into it the identity[1],

$$1 = \int d^n \omega \Delta_\omega \prod_{i=1}^{n} \delta(<A - A_0^\omega, f_\omega^{(i)}>), \qquad (2.5)$$

where n is the number of collective coordinates, $<X, Y> \equiv \int d^4 x X(x) Y(x)$ is just the ordinary inner product for functions[2], $f^{(i)}$ are some constraint functions that in general depend on ω. Notice that A_0, being the solution to (2.2), is independent of the choice of constraints. The Jacobian Δ_ω is given to the leading order in g by,

$$\Delta_\omega = \left| \det_{ij} (< \frac{\partial A_0^\omega}{\partial \omega^{(i)}}, f_\omega^{(j)}>) \right|. \qquad (2.6)$$

The δ-function in (2.5) can be further massaged using

$$\delta(x) = \lim_{\alpha \to 0} \frac{1}{\sqrt{2\pi\alpha}} \exp\left(-\frac{1}{2\alpha} x^2\right). \qquad (2.7)$$

This gives

$$Z = \int d^n \omega Z_\omega, \qquad (2.8)$$

and

$$Z_\omega \sim e^{-S(A_0^\omega)} \Delta_\omega \lim_{\alpha \to 0} \frac{1}{(2\pi\alpha)^{\frac{n}{2}}} \int [\mathcal{D}(\delta A)] \exp(-\frac{1}{2} \delta A \; H \; \delta A)$$
$$\sim e^{-S(A_0^\omega)} \Delta_\omega \lim_{\alpha \to 0} \frac{1}{(2\pi\alpha)^{\frac{n}{2}}} \det(H)^{-\frac{1}{2}}, \qquad (2.9)$$

where

$$H = \left. \frac{\delta^2 S}{\delta A^2} \right|_{A_0} + \frac{1}{\alpha} \sum_i f_\omega^{(i)} \times f_\omega^{(i)}. \qquad (2.10)$$

Since the operator $\left. \frac{\delta^2 S}{\delta A^2} \right|_{A_0}$ has zero modes $\frac{\partial A_0^\omega}{\partial \omega^{(i)}}$, we modify the identity

$$\det(M + \sum_i b^i \times b^i) = \det M \det_{ij}(\delta_{ij} + <b^i, M^{-1}b^j>) \qquad (2.11)$$

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[1] Of course for the identity to hold, we have to assume that for any given A, $<A - A_0^\omega, f_\omega^{(i)}> = 0$ has exactly one solution, $\omega^*(A)$. However, because we are taking the semi-classical approximation, this condition in fact only needs to be imposed on A that is fairly close to A_0^ω.

[2] We will be using a general constraint, f, thus introducing generalized inner products is unnecessary.

into

$$\det(M + \sum_i b^i \times b^i) \;=\; \det{}' M \det_{ik} < b^i, a^k > \det_{kl}^{-1} < a^k, a^l > \det_{lj} < a^l, b^j >, \quad (2.12)$$

where M is any given matrix, b is a given set of n vectors, a denotes the n zero modes of M and $\det'$ denotes the nonzero part of the determinant. We finally arrive at

$$Z_\omega \;\sim\; \det_{ij}^{\frac{1}{2}}(< \frac{\partial A_0^\omega}{\partial \omega^{(i)}}, \frac{\partial A_0^\omega}{\partial \omega^{(j)}} >) \det{}'^{-\frac{1}{2}}\left(\frac{\delta^2 S}{\delta A^2}\bigg|_{A_0}\right) e^{-S(A_0^\omega)}. \quad (2.13)$$

This coincides with the result in ref.8 except for a minus sign in their formula, which apparently results from their not taking the absolute value in the counterpart of (2.6). Note that all f–dependences cancel among themselves and as a result, f does not enter the expression at all, just as one would have expected.

If these zero modes are instead quasi-zero modes , which will be denoted ξ, the same derivation still applies. Once A_0 is written down[3], we can again go through (2.5) to (2.10). The only change is that instead of (2.12), we use (2.11) itself. This gives

$$Z \;=\; \int d^n \xi Z_\xi, \quad (2.14)$$

and

$$Z_\xi \;\sim\; \det_{ij}(< \frac{\partial A_0^\xi}{\partial \xi^{(i)}}, f_\xi^{(j)} >) \quad \det_{ij}^{-\frac{1}{2}}(< f_\xi^{(i)}, \left(\frac{\delta^2 S}{\delta A^2}\bigg|_{A_0}\right)^{-1} f_\xi^{(j)} >)$$
$$\det{}^{-\frac{1}{2}}\left(\frac{\delta^2 S}{\delta A^2}\bigg|_{A_0}\right) e^{-S(A_0^\xi)}, \quad (2.15)$$

which differs slightly from the result in ref.8,

$$Z_\xi \;\sim\; \det_{ij}(- < f_\xi^{(i)}, f_\xi^{(j)} >) \quad \det_{ij}^{-\frac{1}{2}}(- < f_\xi^{(i)}, \left(\frac{\delta^2 S}{\delta A^2}\bigg|_{A_0}\right)^{-1} f_\xi^{(j)} >)$$
$$\det{}^{-\frac{1}{2}}\left(\frac{\delta^2 S}{\delta A^2}\bigg|_{A_0}\right) e^{-S(A_0^\xi)}. \quad (2.16)$$

It is worth pointing out that (2.15) should be the exact and whole leading term in the semi-classical expansion. Therefore, it cannot depend on the arbitrary choice of contraints,

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[3] To find A_0 is in fact where the real difficulty lies. We will elaborate on this later.

which apparently determines the exact form for A_0^ξ and in turn $S(A_0^\xi)$. This implicit f-dependence in the exponent has to be cancelled by the dependence in the pre-exponential factor. This fact can be used as a consistency check on (2.15). Notice that A_0^ξ should remain unchanged while we make the substitution $f_\xi(x) \rightarrow c(\xi)f_\xi(x)$, where $c(\xi)$ is a constant in spacetime. Under this "transformation", (2.15) is invariant, while (2.16) obviously is not.

Some readers may be alarmed by the fact that our constraint function f has a dependence on ξ, and thus does not divide the entire functional space into well-defined slices. By this we mean that for any given field configuration A, we should find one unique background field A_0^ξ by demanding $< (A - A_0^\xi), f_\xi >= 0$, such that we can consider all the fields A corresponding to the same A_0^ξ as its quantum fluctuation. It is such a consideration that prompts the author of ref.2 to propose global constraints (such as $\int \text{tr}F^3 \propto \rho^{-2}$). Unfortunately, they are nonlinear and make solving for A_0^ξ practically impossible. We circumvent this dilemma by noting that in our semi-classical method, all configurations far away from the valley trajectory A_0^ξ are approximated by the Gaussian form anyway. Thus we only need well-defined slices along a small neighborhood of A_0^ξ, justifying our use of f_ξ.

This is not the whole story yet, because although (2.15) is an elegant expression, it is very difficult to evaluate the pre-exponential factor in practice. One thus uses the exponential part as an approximation. Ideally one would like to choose A_0^ξ (or equivalently f_ξ) so that the pre-exponential factor in (2.15) is independent of ξ, but this again is a hopeless task. The authors of refs. 3 and 4 pursued a "natural constraint",

$$f_\xi \propto \frac{\partial A_0^\xi}{\partial \xi},\qquad(2.17)$$

which would simplify the expression a little[4], but does not necessarily minimize the contribution from the pre-exponential factor. They did venture out to define a "generalized" inner product, but only after using a spherical ansatz to reduce the Yang-Mills lagrangian to that of a simple quantum mechanical system for which one has much better physical intuition. The result still turns out to be undesirable, underscoring the importance of following our systematic approach. We will elaborate on this fact in Section 4. Readers looking for greater details will find them in ref.6.

It turns out that our best course of action is to make the best use of this freedom in choosing the constraint. This would require that we isolate the f-dependent part of

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[4] The effects of equation (2.17) will be evaluated at the end of Section 3.

A_0^ξ, make a reasonable guess for it, add back in the f-independent part, and then find the corresponding constraint f. In practice, we first have to identify the quasi-zero modes that need to be treated. Next, we write down the set of criteria for desirable solutions A_0^ξ. We then try to find a solution which satisfies all the criteria. These tasks are far from trivial, but still much more manageable than to solve A_0^ξ for a given f_ξ. To find the corresponding constraint function f_ξ for a given A_0^ξ is relatively easy because basically we are looking for such an f_ξ that

$$< \frac{\delta S}{\delta A}\bigg|_{A_0^\xi}, \delta A > = \ 0, \text{ for any } \delta A \text{ satisfying } < f_\xi, \delta A >= 0. \tag{2.18}$$

Apparently we should always choose one of the f's to be $\frac{\delta S}{\delta A}\big|_{A_0^\xi}$. The choice for the rest of the f's is again arbitrary and does not affect the final outcome. Notice that this represents a dramatic change from the way we treat the field equation (2.2). We need to solve (2.2) for A_0. We can also do the same with (2.18) if given f, but the alternative way is to use it to identify the corresponding f for any given A_0. As explained above, this is much easier.

We also want to reemphasize that unlike in the case of zero modes, there is no such thing as an "exact solution" for quasi-zero modes a priori. All trajectories will yield the same correct answer in (2.15). Only when we ignore part of (2.15) do we have a preference for those which minimize the effects of the ignored part. Even then we still cannot say which solution is best exactly because of the uncertainty introduced by discarding the incalculable terms. But for problems that require only a qualitative or partially quantitative understanding, the valley method remains a useful tool. We demonstrate our systematic approach for the Yang-Mills-Higgs system in the following sections. Detailed discussions for the pure Yang-Mills theory can be found in ref.6.

3. The Constrained Instanton in the Yang-Mills-Higgs Theory

As mentioned earlier, in the instanton number $Q = 1$ sector the lowest lying configuration of the pure Yang-Mills action is an instanton with 8 zero-modes. If we add Higgs and its vacuum expectation value into the system, the scale invariance is broken, and the ρ direction is lifted to a quasi-zero mode. This can be seen by observing the effect the rescaling $A(x) \rightarrow aA(ax)$, $\phi(x) \rightarrow \phi(ax)$ has on the action[1,2]. The constrained instanton in the Yang-Mills-Higgs theory is thus obviously a valley trajectory parametrized by ρ.

After identifying the valley direction, our next task is to set up criteria for the desirable solution $\{A_0^\rho, \phi_0^\rho\}$. Guided by our earlier discussion and previous experience with the instanton-antiinstanton pair in the pure Yang-Mills theory[6], we find the following,

1) $\{A_0^\rho, \phi_0^\rho\}$ belongs in the $Q = 1$ sector.

2) $\{A_0^\rho, \phi_0^\rho\}$ has easily identifiable instanton parameters, and covers the entire 8-dimensional parameter space spanned by these zero- and nonzero-modes.

3) $\{A_0^\rho, \phi_0^\rho\}$ conforms to all constraint-independent results, which can usually be obtained by taking the parameters to certain limits.

4) $\{A_0^\rho, \phi_0^\rho\}$ respects the symmetries of the theory.

Just like in the pure Yang-Mills $I\bar{I}$ case, $Cri.3$ is the most restrictive and therefore the most useful for our purpose. There is a subtlety here however. The Higgs vacuum expectation value $\langle\phi\rangle$, which explicitly breaks the ρ invariance in the action, should have an interesting interplay with ρ when we take the limits. We now examine this in detail.

Consider the rescaled Yang-Mills-Higgs action

$$
S \;=\; \frac{1}{g^2} \int d^4x \left\{ \frac{1}{2} tr F^2 + \kappa \left[\left| D\phi \right|^2 + \frac{1}{4} \left(\left| \phi \right|^2 - \langle\phi\rangle^2 \right)^2 \right] \right\}, \tag{3.1}
$$

where κ is the ratio between gauge and Higgs couplings. We then introduce the following ansatz[5]

$$
A_\mu(x) \;=\; \frac{x^\nu \Lambda_{\mu\nu}}{x^2} B(r), \quad \phi(x) \;=\; (1 - H(r))\langle\phi\rangle, \tag{3.2}
$$

where $r = \left| x \right|$, $\Lambda_{\mu\nu} = \frac{1}{4i}(\sigma_\mu \bar{\sigma}_\nu - \sigma_\nu \bar{\sigma}_\mu)$ and $\sigma_\mu = (\vec{\sigma}, i)$, $\bar{\sigma}_\mu = (\vec{\sigma}, -i)$. The only dimensional variables in the theory are r, ρ, $\langle\phi\rangle$ and the renormalization scale μ. Knowing that effects of μ can always be absorbed into the renormalization of the coupling constant, the dimensionless B and H can only depend on the variables, $\frac{\rho}{r}$ and $\langle\phi\rangle r$. Imagine the double

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[5] The use of this ansatz is by no means essential to our reasoning, but will serve to simplify the presentation greatly.

expansions of B and H with respect to these two parameters[6]

$$
\begin{aligned}
B(\frac{\rho}{r}, \langle\phi\rangle r) &= B_0(\frac{\rho}{r}) + (\langle\phi\rangle r)^2 B_1(\frac{\rho}{r}) + \cdots \\
&= (\frac{\rho}{r})^2 B^0(\langle\phi\rangle r) + (\frac{\rho}{r})^4 B^1(\langle\phi\rangle r) + \cdots, \\
H(\frac{\rho}{r}, \langle\phi\rangle r) &= H_0(\frac{\rho}{r}) + (\langle\phi\rangle r)^2 \ln\langle\phi\rangle r H_1(\frac{\rho}{r}) + \cdots \\
&= (\frac{\rho}{r})^2 H^0(\langle\phi\rangle r) + (\frac{\rho}{r})^4 H^1(\langle\phi\rangle r) + \cdots,
\end{aligned}
\tag{3.3}
$$

in which B_i and H_i will be considered "row vectors", and B^j, H^j "column vectors". Notice that these expansions are not analytic. They contain logarithms. The exact forms of the expansions can be inferred from expanding B_0, H_0 with respect to $\frac{\rho}{r}$, and B^0, H^0 with respect to $\langle\phi\rangle r$, which are exactly calculable. We now compute the leading row by taking the r (or equivalently $\langle\phi\rangle$) $\to 0$ limit and the leading column by taking $r \to \infty$. We have the following constraint-independent results[2,9]

$$
\begin{aligned}
B_0 &= \frac{2\rho^2}{r^2 + \rho^2}, \\
H_0 &= 1 - \left(\frac{r^2}{r^2 + \rho^2}\right)^{\frac{1}{2}},
\end{aligned}
\tag{3.4}
$$

and

$$
\begin{aligned}
B^0 &= m_W^2 r^2 K_2(m_W r), \\
H^0 &= \frac{m_H r}{2} K_1(m_H r),
\end{aligned}
\tag{3.5}
$$

where $m_W = \sqrt{\frac{\kappa}{2}}\langle\phi\rangle$ and $m_H = \langle\phi\rangle$ are the gauge boson and Higgs masses respectively, and K_ν is the modified Bessel function of the second kind, with the following small y behaviors,

$$
\begin{aligned}
K_1(y) &\sim \frac{1}{y} + \mathcal{O}\left(y \ln y\right) \\
K_2(y) &\sim \frac{2}{y^2} - \frac{1}{2} + \mathcal{O}\left(y^2 \ln y\right).
\end{aligned}
\tag{3.6}
$$

It was shown in ref.2 that all higher order terms have dependence on the constraint we choose. Since by our valley method, all constraint-dependent terms can be freely chosen

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[6] If we require that at least for some values of r, both $\frac{\rho}{r}$ and $\langle\phi\rangle r$ are small, we will have to assume that $\rho\langle\phi\rangle$ is small, but we can always push our result to the $\rho\langle\phi\rangle \geq 1$ region later. For $\rho\langle\phi\rangle \gg 1$, the renormalized coupling constant becomes large, invalidating our semi-classical approximation, but contributions from that part of phase space should be highly suppressed anyway.

first and used to determine the corresponding constraint later, we can now fill in the blank in any way we like. How does one fill in the blank in a matrix M when the first row M_0 and the first column M^0 are given? Well, the simplest choice seems to be

$$M \equiv \frac{1}{M_0^0} M^0 \times M_0, \tag{3.7}$$

which corresponds to the following choice

$$B(r) = \frac{r^2 \rho^2 m_W^2}{r^2 + \rho^2} K_2(m_W r),$$

$$H(r) = \left[1 - \left(\frac{r^2}{r^2 + \rho^2} \right)^{\frac{1}{2}} \right] m_H r K_1(m_H r), \tag{3.8}$$

or

$$A_\mu(x) = \frac{x_\nu \Lambda_{\mu\nu} \rho^2 m_W^2}{x^2 + \rho^2} K_2(m_W |x|),$$

$$\phi(x) = \left\{ 1 - \left[1 - \left(\frac{x^2}{x^2 + \rho^2} \right)^{\frac{1}{2}} \right] m_H |x| K_1(m_H |x|) \right\} \langle \phi \rangle. \tag{3.9}$$

Of course, there are actually infinitely many ways to form a matrix with only one row and one column fixed. For any background matrix M_B, define $\Delta M^0 = M^0 - M_B^0$ and $\Delta M_0 = M_0 - M_{B0}$. It is obvious that the choice

$$M_B + \frac{1}{\Delta M_0^0} \Delta M^0 \times \Delta M_0 \tag{3.10}$$

would work just as well[7]. This would seem to indicate that there is nothing special about (3.9). For example, we may just as well set all constraint-dependent terms to zero, which corresponds to

$$A_\mu(x) = \frac{x_\nu \Lambda_{\mu\nu}}{x^2} \left[\rho^2 m_W^2 K_2(m_W |x|) - \frac{2\rho^4}{x^2 (x^2 + \rho^2)} \right],$$

$$\phi(x) = \left[\left(\frac{x^2}{x^2 + \rho^2} \right)^{\frac{1}{2}} + \frac{\rho^2}{2x^2} - m_H r |x| K_1(m_H |x|) \right] \langle \phi \rangle. \tag{3.11}$$

Strangely (3.11) has the wrong asyptotic behaviors, and cannot be considered a good solution. We seem to have a paradox at hand. Of course the cause of this paradox is

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[7] In fact, the converse is also true. For any matrix M, we can always find a certain (but not unique) M_B such that the expression (3.10) equals M.

that we have lied. We are not really free to choose all the higher order terms in (3.3) even though the dependence on the constraint is genuine. This is because there is an implicit requirement that these higher order terms cannot dominate over the preceding terms. To put it more precisely, the ratio $\frac{B_{i+1}}{B_i}$ better be bounded above for all values of r. The same goes for $\frac{B^{i+1}}{B^i}$, $\frac{H_{i+1}}{H_i}$ and $\frac{H^{i+1}}{H^i}$. In effect, (3.9) corresponds to setting all these ratios to constants. Now it is no longer trivial to write down alternatives to (3.9), although they certainly exist in principle. In fact, we will now introduce one of these alternative expressions which is more elegant, especially when used to construct the instanton-antiinstanton pair solution in the next section.

$$
\begin{aligned}
A_\mu(x) &= \frac{x_\nu \Lambda_{\mu\nu} \rho^2 m_W^2 K_2(m_W|x|)}{x^2 + \rho^2 m_W^2 x^2 K_2(m_W|x|)/2}, \\
\phi(x) &= \left(\frac{x^2}{x^2 + \rho^2 m_H |x| K_1(m_H|x|)} \right)^{\frac{1}{2}} \langle \phi \rangle.
\end{aligned}
\tag{3.12}
$$

This is what we consider the best valley solution for a constrained instanton. One can easily check that all criteria are satisfied.

Now that we have found the valley configuration, we will proceed to find the corresponding constraint. As explained in the previous section, this is rather straightforward, especially since there is only one quasi-zero mode. f_ρ is none other than $\frac{\delta S}{\delta A}\big|_{A_0^\rho}$, where A stands for both A_μ and ϕ, and A_0^ρ is (3.9). Substituting this into (2.15), we have

$$
\begin{aligned}
Z &= \int d\rho \ Z_\rho \\
&\sim \int d\rho \ < \frac{\partial A_0^\rho}{\partial \rho}, \frac{\delta S}{\delta A}\Big|_{A_0^\rho} > \ < \frac{\delta S}{\delta A}\Big|_{A_0^\rho}, \left(\frac{\delta^2 S}{\delta A^2}\Big|_{A_0} \right)^{-1} \frac{\delta S}{\delta A}\Big|_{A_0^\rho} >^{-\frac{1}{2}} \\
&\quad \det^{-\frac{1}{2}} \left(\frac{\delta^2 S}{\delta A^2}\Big|_{A_0} \right) e^{-S(A_0^\rho)} \\
&= \int d\rho \ \frac{dS}{d\rho} \ \left| \frac{\delta S}{\delta A}\Big|_{A_0^\rho} \right|^{-1} \det''^{-\frac{1}{2}} \left(\frac{\delta^2 S}{\delta A^2}\Big|_{A_0} \right) e^{-S(A_0^\rho)},
\end{aligned}
\tag{3.13}
$$

where $\det''$ is the determinant taken in the slices defined by the constraint[8].

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[8] Here we make a comparison with the nonlinear constraint $\delta \left(\int d^4x \left[\mathcal{O}(A) - \mathcal{O}(\rho) \right] \right)$ used in ref.2. To leading order in g, this reduces to $\delta(< \frac{\delta \mathcal{O}}{\delta A}\big|_{A_0}, \delta A >)$. Comparing this with (2.18), we see that our constraint corresponds to choosing $\mathcal{O}$ to be the lagrangian. Unfortunately, this implies

It would also be interesting to see what we could get by imposing the "natural constraint" condition (2.17). This would mean abandoning (3.9) and instead pursuing the solution A_0 which satisfies

$$\left.\frac{\delta S}{\delta A}\right|_{A_0^\rho} \propto \frac{\partial A_0^\rho}{\partial \rho}. \tag{3.14}$$

Let's suppose for a moment that such a solution could indeed be found. Equation (3.13) will then be slightly simplified. We have

$$Z \sim \int d\rho < \frac{\partial A_0^\rho}{\partial \rho}, \frac{\partial A_0^\rho}{\partial \rho} >^{\frac{1}{2}} \ \mathrm{det}''^{-\frac{1}{2}}\left(\left.\frac{\delta^2 S}{\delta A^2}\right|_{A_0}\right) e^{-S(A_0^\rho)}, \tag{3.15}$$

which is very similar to (2.13), our formula for exact zero modes. The first term in the integral can be easily calculated if we are given A_0^ρ. However, the second term, which is probably beyond our reach, also has a nontrivial dependence on ρ which is not necessarily negligible. Since in reality we cannot even solve for A_0^ρ under (3.14), we have nothing to gain by taking this approach.

4. Baryon Number Violation and the Instanton-Antiinstanton Pair

It was first suggested by Ringwald[10] and later Espinosa [9] that anomalous baryon number violation in the standard model might be observable in high-energy colliders. They, and other authors later, suggested that at nonzero energy, the 't Hooft suppression factor[1] $\exp(-4\pi/\alpha)$ for this process is replaced by

$$\sigma \sim \exp\left\{\left(\frac{4\pi}{\alpha}\right) F_{hg}\left(\left(\frac{E}{E_{sphal}}\right)^{\frac{2}{3}}\right)\right\}, \tag{4.1}$$

---

that all field configurations in a slice defined by this constraint have the same action. Therefore we shouldn't be able to solve for A_0, which is by definition the configuration with the lowest action in the slice, unless we switch back to our linear version of the constraint. Naively, if we ignore this fact and go through the formal maneuver of algebra, we find, for example, the equation for B_1 is

$$\left.\frac{\delta^2 S}{\delta A^2}\right|_{B_0,H_0} (B_1 - \sigma_1 B_0) = 0,$$

where σ_1 is the leading term in the lagrangian multiplier. Since our solution (3.8) corresponds to

$$B_1 \propto B_0,$$

it is satisfactory.

where $E_{sphal} \sim m_W/\alpha$ is the sphaleron scale, and F_{hg} is the so-called "holy grail" function which increases from -1 at zero energy to values presumably closer to 0 near E_{sphal}. These pioneering works are based on single-instanton calculations and do not take into account corrections from the incoming particles, the outgoing particles and multiple instanton effects[11,12]. We have very little to say about the incoming particle corrections[13,14], and will concentrate our attention on the contributions coming from the instanton-antiinstanton pair[5,15,16], which has been shown to sum up the outgoing particle corrections perturbatively to all orders of $\frac{E}{E_{sphal}}$[17]. Little progress has been made so far on multiple instanton effects due to the difficulty of finding expressions corresponding to multiple instantons and antiinstantons. Our earlier paper[6] gives these expressions for the pure Yang-Mills theory. It would be interesting to carry over the results to the Yang-Mills-Higgs theory, but the subject is really beyond the scope of this paper and will have to await future efforts.

The idea that puts the valley method and the baryon number violation together is the following. Instead of calculating the $\langle 2 \rightarrow \text{any} \rangle$ cross section one by one, one applies the optical theorem and computes the imaginary part of the forward $\langle 2 \rightarrow 2 \rangle$ amplitude. Since we are only interested in the anomalous part of the total amplitude, the idea is to consider the $\langle 2 \rightarrow 2 \rangle$ amplitude in the presence of an instanton-antiinstanton pair background. This leap of faith turns out to be valid, at least in the sense that it provides the analytic continuation of F_{hg} from its low-energy limit[17]. The actual computation is still difficult. Since so little was known about the Yang-Mills-Higgs $I\bar{I}$ configuration at the time, these authors[5] resorted to using the action S_{Yung} of an available pure Yang-Mills valley, together with the leading contribution from the Higgs sectors of a single constrained Yang-Mills-Higgs instanton and an antiinstanton. They therefore had

$$\sigma \sim \text{Im} \int dR \, d\rho_1 \, d\rho_2 \, \exp[ER - \pi^2(\rho_1^2 + \rho_2^2)\langle\phi\rangle^2 - S_{\text{Yung}}], \qquad (4.2)$$

where R is the separation between I and $\bar{I}$, and ρ_1 and ρ_2 are the sizes of the instanton and the antiinstanton respectively. The first term in the exponent is the naive initial state phase factor. After the change of variables,

$$\begin{aligned}
\theta &= R/\rho, & \bar{\rho}_i &= \frac{1}{2}g\langle\phi\rangle\rho, \\
\bar{E} &= gE/8\pi^2\langle\phi\rangle, & \bar{S} &= (g^2/16\pi^2)S_{\text{Yung}},
\end{aligned} \qquad (4.3)$$

they proceeded with a saddle-point analysis. Assuming $\rho_1 = \rho_2 \equiv \rho$ at the saddle point, they arrived at

$$\bar{\rho} = \bar{E}\theta, \quad \bar{S}'(\theta) = \bar{E}\bar{\rho}. \tag{4.4}$$

Eliminating $\bar{\rho}$ then leaves

$$\bar{E}^2\theta = \bar{S}'(\theta). \tag{4.5}$$

Because S_{Yung} is exactly known, (4.5) can be solved either graphically or numerically. For small E, the large θ expansion of $\bar{S}$,

$$\bar{S}(\theta) \rightarrow 1 - 6/\theta^4 + O(\theta^{-6}), \tag{4.6}$$

can be used to solve for $\theta(\bar{E})$. Substituting the result back into (4.2), they had

$$F_{hg} \sim -1 + \frac{9}{8}\left(\frac{E}{E_{sphal}}\right)^{\frac{4}{3}} + \cdots, \tag{4.7}$$

in agreement with the results previously derived using other methods.

As energy increases towards a finite critical energy E_{crit}, which is determined by the slope of $\bar{S}'$ at $\theta = 0$, the solution $\theta(\bar{E})$ tends to zero, hence F_{hg} approaches zero as well. This seems to indicate that E_{crit} is the energy at which anomalous processes are no longer suppressed. Simple algebra shows that for small θ's,

$$\bar{S}(\theta) = \frac{6}{5}\theta^2 - \frac{4}{5}\theta^3 + \frac{9}{35}\theta^4 + \cdots. \tag{4.8}$$

E_{crit} is determined by the coefficient of the first term,

$$E_{crit} = \sqrt{\frac{12}{5}}E_{sphal}. \tag{4.9}$$

It is probably not surprising that this analysis was greeted with some skepticism. Some authors argued that for small values of θ, the valley configuration should be considered a quantum fluctuation of the trivial vacuum. Therefore it contributes to the non-anomalous cross section instead of the anomalous one. This is a valid concern, and also a source of great confusion in the literature. (There have been some papers which argue that cuts or poles exist to separate the anomalous part from the non-anomalous one in the valley approach.) Our view on this issue is the following. There are three relevant versions of the optical theorem. The general version is all-inclusive, and thus mixes the anomalous

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part with the non-anomalous one, and the multi-instanton contributions with the single-instanton ones. Obviously it is not useful for our purpose. There is also a perturbative optical theorem based on Cutkosky's cutting rules[18,19]. It was originally applied to the trivial-vacuum background, and simply stated that the optical theorem still holds when restricted to perturbative non-anomalous cross-section. As far as the valley method is concerned, this corresponds to expanding $\bar{S}(\theta)$ in θ at the origin. Therefore eq.(4.8) just gives the non-anomalous result in ordinary perturbation theory. The third version of the optical theorem was developed specifically for the anomalous processes[17]. These authors applied the cutting rules to a widely-separated $I\bar{I}$ background configuration. They showed that the anomalous cross-section can also be related to the imaginary part of the $\langle 2 \rightarrow 2 \rangle$ amplitude in a perturbation series. The expansion parameter is $1/\theta$, and the expansion point is at $1/\theta = 0$(see (4.6)). In terms of F_{hg}, this is equivalent to an expansion in powers of E. Clearly, both the non-anomalous and the anomalous cross-sections calculated this way are perturbative in nature, and are valid only within the range of validity of the expansion series, (4.8) and (4.6) respectively. Since the two do not intersect each other, there is no paradox.

Before we proceed to derive an expression for the instanton-antiinstanton pair, which in principle will allow us to compute the anomalous cross-section perturbatively to all orders in E, let's first assume that for some unknown reason, the analysis leading to eq.(4.9) has its merits and is worth refining. In fact, several authors have made the extra effort to criticize (4.9) on technical grounds. Unfortunately these criticisms themselves are often prone to errors. For example, the author of ref.11 argued that the freedom in choosing the constraint allows one to find valleys which give different $\bar{S}(\theta)$'s. Since E_{crit} is determined by $\bar{S}''$, we won't have a definite prediction for E_{crit}. To make matters worse, the analysis leading to (4.9) assumes that $\bar{S}''' < 0$. When $\bar{S}'''$ is made positive, the solution "bifurcates". These problems were then incorrectly attributed to the unknown incoming particle contributions.

Based on our previous discussion, we know that these inconsistencies should be resolved within the scope of (2.15). A valley is called a valley because it rises slower than quadratic. Therefore if one encounters positive third derivatives, it is obvious that the valley trajectory has been so badly chosen that previously discarded pre-exponential factors now dominate. In fact, the third derivative of $\bar{S}$ should always be zero. It is the fourth

derivative that will remain negative. This is the result of a Z_2 symmetry[9] , by which I mean $\bar{S}$ should be invariant under $\theta \to -\theta$. Therefore Yung's $I\bar{I}$ solution does not even qualify for a good valley for the pure Yang-Mills theory (Remember $Cri.3$?). Fortunately we have found another better solution $A_{I\bar{I}}^{\mathrm{YM}}$ which we shall derive later in (4.16). It gives

$$\bar{S}(\theta) = \frac{6}{5}\theta^2 - \frac{33}{35}\theta^4 + \cdots. \tag{4.10}$$

Notice that the transformation used in ref.11 to demonstrate the possibility of changing $\bar{S}''$, conflicts with the Z_2 symmetry mentioned above. Therefore it cannot be applied on our $A_{I\bar{I}}$ to generate other solutions.

We now begin our quest for the expression $A_{I\bar{I}}^{\mathrm{YMH}}$ for the Yang-Mills-Higgs instanton-antiinstanton pair, which should enable us to calculate not only h but also the whole action to all orders in ρ. We will start by reviewing the derivation for $A_{I\bar{I}}^{\mathrm{YM}}$. Notice that classically the 4-dimensional pure Yang-Mills theory has the conformal group as its symmetry. Therefore those 15 generators of the conformal group should correspond to zero-modes. Unfortunately the analysis is more complicated than one would have expected because some of them become gauge zero-modes sometimes,*i.e.* some conformal transformations can be undone by gauge transformations. For the $I\bar{I}$ configuration, we start with 16 parameters, consisting of 8 positions, 2 sizes and 6 phases. After some tedious examination, we find that all of them correspond to zero-modes except for a mixed parameter $z = (R^2 + \rho_1^2 + \rho_2^2)/2\rho_1\rho_2$, which can be interpreted as the separation between $I\bar{I}$, and one relative phase, which we will ignore for now. Just like the constrained instanton in the Yang-Mills-Higgs theory we discussed in Section 3, we first write down the four criteria,

1) $A_{I\bar{I}}^{\mathrm{YM}}$ belongs in the $Q = 0$ sector.
2) $A_{I\bar{I}}^{\mathrm{YM}}$ has easily identifiable instanton parameters, and covers the entire 16-dimensional parameter space spanned by the zero- and nonzero-modes, although we will not worry about the phases until later.
3) $A_{I\bar{I}}^{\mathrm{YM}}$ conforms to all constraint-independent results. In this case, we in fact require that

---

[9] Remember that θ corresponds to the quasi-zero mode $\left|\vec{R}\right|$, where $\vec{R}$ is the displacement between I and $\bar{I}$. $\vec{R}$ has four degrees of freedom, the other three being zero-modes of the theory. If we require S to be a smooth function of $\vec{R}$, in particular at the origin, we find that we need to impose the Z_2 symmetry.

3.1) $A^{\rm YM}_{I\bar{I}}$ becomes the sum of an instanton and an antiinstanton for large z.

3.2) $A^{\rm YM}_{I\bar{I}}$ vanishes as $z \to 0$.

4) $A^{\rm YM}_{I\bar{I}}$ respects the symmetries of the theory.

Obviously the most straightforward way to satisfy $Cri.2$ and $Cri.3.1$ is to write $A^{\rm YM}_{I\bar{I}}$ as the linear combination of an instanton and an antiinstanton. Just putting the two objects together doesn't work well though, as many earlier researchers discovered to their dismay. The difficulty lies in the small z behavior. It is harder than it looks to make such a linear expression vanish, especially if one insists on having the two objects in the same gauge, which most of these authors did. An interesting phenomenon which first appeared in $A_{\rm Yung}$ (although no one realized its importance then) is that if one of the (anti)instantons is set in the regular gauge and the other in the singular gauge, the sum of the two becomes pure gauge at $z = 0$, $i.e.$

$$\frac{2x^\nu \Lambda_{\mu\nu}}{x^2 + \rho^2} + \frac{2\rho^2 x^\nu \Lambda_{\mu\nu}}{x^2(x^2 + \rho^2)} = \frac{2x^\nu \Lambda_{\mu\nu}}{x^2}, \tag{4.11}$$

This pure gauge configuration can then be transformed into the trivial vacuum by the gauge transformation

$$A \to g_0^{-1} A g_0 + g_0^{-1} dg_0, \tag{4.12}$$

where $g_0(x) = \sum_\nu x^\nu \sigma_\nu / |x|$. This algebraic coincidence strongly suggests that a linear combination of an antiinstanton at the origin in the regular gauge and an instanton at $x = \vec{R}$ in the singular gauge (or the other way around) should be considered, $i.e.$

$$A_\mu = \frac{2x^\nu \Lambda_{\mu\nu}}{x^2 + \rho_1^2} + \frac{2\rho_2^2 (x^\nu - R^\nu) \Lambda_{\mu\nu}}{(x - \vec{R})^2[(x - \vec{R})^2 + \rho_2^2]}, \tag{4.13}$$

where $\vec{R}$ is a 4-vector. This expression will automatically satisfy $Cri.1, 2$ and 3. $Cri.4$ is trickier. It implies that all $\{R, \rho_1, \rho_2\}$ of the same conformal class ($i.e.$ those that can related to each others by conformal transformations) should give the same action. Unfortunately, this is not true with (4.13). There are three quasi-zero modes, $R = |\vec{R}|$, ρ_1 and ρ_2, while we want only one. The simplest way to conform to the conformal symmetry is the following. We can choose a one-dimensional slice in the three dimensional parameter space spanned by R, ρ_1 and ρ_2, use (4.13) on this slice only, and project the solution on this slice to the rest of the parameter space using the conformal group. If one chooses the slice to be

$$\vec{R} = 0, \qquad \rho_1 \rho_2 = 1, \tag{4.14}$$

one recovers Yung's $I\bar{I}$ solution. If one chooses instead

$$\rho_1 \; = \; \rho_2 \; = \; 1, \tag{4.15}$$

we have the solution which we shall refer to as $A_{I\bar{I}}^{\mathrm{YM}}$ exclusively from now on,

$$A_\mu \; = \; \frac{2x^\nu \Lambda_{\mu\nu}}{x^2 + 1} + \frac{2(x^\nu - R^\nu)\Lambda_{\mu\nu}}{(x - \vec{R})^2[(x - \vec{R})^2 + 1]}. \tag{4.16}$$

Our choice has a few advantages over Yung's solution. Besides satisfying the Z_2 symmetry[10] [11] mentioned earlier, it directly corresponds to the relevant quasi-zero modes in its Yang-Mills-Higgs counterpart, as we shall show below.

In the Yang-Mills-Higgs system, R, ρ_1 and ρ_2 are all quasi-zero modes. Therefore we shall look for the analog of (4.13)[12]. Notice that (3.12) is written in the singular gauge. For an constrained antiinstanton in the regular gauge, we take the conjugate ($\Lambda \to \bar{\Lambda}$) of (3.12) and apply the gauge transformation (4.12) on it. We have

$$
\begin{aligned}
A_\mu(x) \; &= \; \frac{2x_\nu \Lambda_{\mu\nu}}{x^2 + \rho^2 m_W^2 x^2 K_2(m_W|x|)/2}, \\
\phi(x) \; &= \; \left(1 + \frac{\rho^2 m_H K_1(m_H|x|)}{|x|}\right)^{-\frac{1}{2}} \frac{x^\nu \sigma_\nu}{|x|}\langle\phi\rangle.
\end{aligned}
\tag{4.17}
$$

Now clearly we need only to put eq.(4.17) and eq.(3.12) together for the gauge field,

$$
\begin{aligned}
A_\mu(x) \; = \; &\frac{2x_\nu \Lambda_{\mu\nu}}{x^2 + \rho_1^2 m_W^2 x^2 K_2(m_W|x|)/2} \\
&+ \frac{(x_\nu - R_\nu)\Lambda_{\mu\nu}\rho_2^2 m_W^2 K_2(m_W|x - R|)}{(x - R)^2 + \rho_2^2 m_W^2 (x - R)^2 K_2(m_W|x - R|)/2},
\end{aligned}
\tag{4.18}
$$

---

[10]   As mentioned earlier, this is the Z_2 symmetry of $R \to -R$. The slice (4.15) has been assumed. If we choose to work on the (4.14) slice, there is another Z_2 which corresponds to $\rho_1/\rho_2 \to \rho_2/\rho_1$. It turns out that these two Z_2 symmetries are in fact conformally equivalent, *i.e.* one implies the other if the solutions on the two slices are related to each other by conformal projection.

[11]   There is another Z_2 symmetry one can impose. This is the space-time reflection symmetry of the lagrangian density for $I\bar{I}$ of identical sizes. Unfortunately neither our solution nor Yung's respect this symmetry. We are not sure whether this constitutes a serious problem.

[12]   Unlike the ρ valley we dealt with in the previous section, the R valley has other solutions which one can write down easily[6]. This doesn't invalidate our results, however. These alternative valley trajectories always exist, no matter we can find them or not.

but what do we do with the Higgs? By examining the term $\left|D\phi\right|^2$ in the action (3.1), we find that ϕ has to have the same winding number as A_μ at infinity, *i.e.*

$$\lim_{x\to\infty} \phi(x) \ \rightarrow \ \frac{x^\nu \sigma_\nu}{|x|} \langle\phi\rangle. \tag{4.19}$$

We shall choose

$$\phi(x) \ = \ \left[1 + \frac{R^2}{1+R^2}\left(\frac{\rho^2 m_H K_1(m_H|x-R|)}{|x-R|} + \frac{\rho^2 m_H K_1(m_H|x|)}{|x|}\right)\right]^{-\frac{1}{2}} \frac{x^\nu \sigma_\nu}{|x|}\langle\phi\rangle. \tag{4.20}$$

Notice that $\phi = 0$ at the centers of the (anti)instantons, which are not where the fermion level crossing occurs as calculated in ref.12. This does not seem to be a problem, because unlike the anthors of ref.12, we do not believe there to be any definite connection between fermion level crossing and a vanishing Higgs VEV.

There is a caveat we have to caution the readers of. The Z_2 symmetry that disqualifies Yung's solution earlier also haunts our $A_{I\bar{I}}^{\mathrm{YMH}}$ in the ρ_1/ρ_2 direction. Fortunately in most cases we are interested in, the saddle point lies in the symmetric configurations, *i.e.* $\rho_1 = \rho_2$. Nonetheless, anyone who is interested in performing calculations based on our $A_{I\bar{I}}^{\mathrm{YMH}}$ solution should always be aware of this limitation.

What is the action of a widely separated YMH $I\bar{I}$ pair as described by eq.(4.18) and eq.(4.20)? The leading interaction term should contain exponential factors: $e^{-M_W r}$ in the gauge part and $e^{-M_H r}$ in the Higgs part. Since $M_W < m_H$, we should concentrate on the gauge interaction, which can be calculated following the derivation of the leading YM $I\bar{I}$ interaction in ref.20. Using the large $M_W r$ approximation for the modified Bessel function,

$$K_2(M_W r) \ \sim \ \left(\frac{\pi}{2M_W r}\right)^{\frac{1}{2}} e^{-M_W r}, \tag{4.21}$$

we find,

$$S_{\mathrm{gauge}} \ \sim \ \frac{16\pi^2}{g^2}\left[1 - 3\rho^4\left(\frac{\pi M_W^3}{2r^5}\right)e^{-M_W r}\right]. \tag{4.22}$$

Replacing S_{Yung} with S_{gauge} in eq.(4.2) then leads to

$$F_{hg} \ \sim \ -1 + \mathcal{O}\left(\frac{E}{M}ln\frac{M}{E}\right), \tag{4.23}$$

which is typical of barrier penetrations and coincides with the results in ref.21, but contradicts the commonly used expansion of F_{hg} in $(E/M)^{\frac{2}{3}}$.

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Of course, the interaction term in eq.(4.22) is nonperturbative, thus the perturbative optical theorem does not relate the valley result (4.23) to the inclusive $\langle 2 \rightarrow 2 \rangle$ cross section. However, we still have to explain the apparent success of the original valley calculation by Khoze and Ringwald[5]. The paradox is the following. The R-term method[22--26] incorporates the large r behavior (which is exponential) of constrained instantons. How can the valley method[5,27] reproduce the correct results with the 't Hooft approximation, which decays as powers of r at large distances? The solution lies in a more careful examination of what the physically relevant limit is. By this, we mean that we should not just take the limit of $E \ll M_{sphal}$ but $M_W \ll E \ll M_{sphal}$ because we are interested in final states with a large number of particles. The corresponding $I\bar{I}$ separation is therefore in the range of $M_W r \ll 1$, where the 't Hooft approximation for (anti-)instantons is valid. The original valley calculation put together a pair of 't Hooft (anti-)instantons and calculated the constraint-independent terms in a series expansion of $M_W r$. This expansion gives the same result as the exact $I\bar{I}$ solution would in the region of interest. Therefore our exact $I\bar{I}$ expression should lead to a closed form for F_{hg} which has the small E expansion (4.7). It is well known[17,22] that constraint dependence starts to manifest itself at order $(E/E_{sphal})^{(8/3)}$. Our formalism should make it easier to find the corresponding constraints for any given valley. At order $(E/E_{sphal})^{(10/3)}$ and higher, the initial state corrections kick in, and the valley method is incomplete.

5. Klinkhamer's New Instanton

Two years ago, Klinkhamer suggested the possible existence of a new constrained "instanton" in the Yang-Mills-Higgs theory[7,29], and its associated sphaleron[30]. The proof of existence was done using the indirect argument that involves the construction of a non-contractible loop under a nonlinear constraint a la Affleck which serves to prevent the collapse of scale. Using the ansatz,

$$
\begin{aligned}
A_\mu &= \tilde{f}(x) \left[\Lambda_{\mu\nu}(x^\nu - R^\nu) + \Lambda_{\tau\nu} x^\nu \frac{1}{2} \mathrm{tr}(\sigma_1 \sigma_\tau \sigma_1 \bar{\sigma}_\mu) \right] \\
\phi &= \tilde{h}(x)\langle\phi\rangle,
\end{aligned}
\tag{5.1}
$$

where σ_1 generates the opposite phases, the new constrained solution was approximated by numerically varying $\tilde{f}, \tilde{h}$ and R to minimize the action under the condition $\tilde{h}(0) = \tilde{h}(R) = 0$.

It is quite obvious that this solution corresponds to a constrained instanton-antiinstanton pair of separation R and equal sizes $\rho_1 = \rho_2 = \rho$. We therefore should look for a way to include in (4.18) a nonzero relative phase $a = a^\nu \sigma_\nu$, where a^ν is a unit 4-vector. We choose to put this phase on the instanton, which is in the singular gauge, so that (4.19) and (4.20) will remain unchanged. We therefore modify (4.18) into

$$
\begin{aligned}
A_\mu(x) \;=\;& \frac{2x_\nu \Lambda_{\mu\nu}}{x^2 + \rho_1^2 m_W^2 x^2 K_2(m_W|x|)/2} \\
&+ \frac{(x_\nu - R_\nu)\frac{1}{2}\mathrm{tr}(\bar{a}\sigma_\tau a\bar{\sigma}_\mu)\Lambda_{\tau\nu}\rho_2^2 m_W^2 K_2(m_W|x-R|)}{(x-R)^2 + \rho_2^2 m_W^2 (x-R)^2 K_2(m_W|x-R|)/2},
\end{aligned}
\tag{5.2}
$$

For the problem at hand, we set $a^\nu = (1,0,0,0)$ to generate the opposite phases needed. The next step is to evaluate the action profile with respect to R, with ρ fixed at various values. If for a given ρ, the minimal action $S(R^*(\rho))$ has a value below twice the one-instanton action, $\{\rho, R^*(\rho)\}$ then gives a new constrained solution. Unfortunately, we are not equipped for the numerical computation required for this task. We present instead the small-R limit of the action for the Yang-Mills instanton-antiinstanton pair with opposite phases,

$$
S(R,\rho) \;\sim\; 2 - \frac{1}{3}\left(\frac{R}{\rho}\right)^2 + \cdots
\tag{5.3}
$$

We do not expect the action profile computed from our (5.2) to differ greatly from the results in ref.7. However, they probably will not be exactly identical either. Afterall, two very different constraints are used, and the resulting approximations do not have to coincide with each other since the pre-exponential factors are again discarded.

Acknowledgements

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